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MOTION OF A SWIRLED FLUID IN THE CORE OF A VERTICAL, TORNADO-LIKE VORTEX

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The motion of a fluid in the core of a vertical, tornado-like vortex has been investigated in the longwave approximation in [1]. Rigorous results were obtained under the assumption that only the azimuthal vorticity component is nonzero in the vortex core.

In the present paper we consider the general case when both azimuthal and vertical vorticity components are nonzero in the core. Although the allowance made for the rotation of the fluid about an axis greatly complicates the analysis, some important features of the motion can be established in this case as well.

1. Statement of the Problem. We examine an inviscid incompressible fluid in a gravity field. The flow is assumed to be steady and rotationally symmetric. We introduce a cylindrical system of coordinates (r, φ, z), where r is the radius, φ is the azimuth angle, z is the axis of symmetry directed opposite to the force of gravity. The region occupied by the fluid is separated into two parts: $r \leq r_0(z)$ is the vortex core and $r > r_0(z)$ is the outer flow. At the core boundary, a jump in the density and in the component of the velocity tangential to the boundary can occur. Length, velocity, and density scales are introduced to transform to dimensionless quantities. The unit of length is the characteristic scale of variation along the z axis; the unit of velocity is the rotational velocity component at z = 0 and $r = r_0$; the unit of density is the density of the outer flow. In this case, the characteristic pressure and acceleration will be equal to unity. We denote by δ the dimensionless r_0 at z = 0. Unless specified otherwise, all quantities are given below in dimensionless form.

The velocity components corresponding to (r, φ, z) are written as (u, v, w); p is the pressure; ρ is the density; and g is the acceleration of gravity. The outer flow is assumed to be known and is specified in a form satisfying the equations of motion

$$u = w = 0, \quad v = \delta/r, \quad p = -\delta^2/(2r^2) - gz. \tag{1.1}$$

It should be noted that (1.1) approximates satisfactorily the outer flow observed in laboratory [2, 3] and atmospheric [4] vortices.

The flow in the vortex core is studied in the long-wave approximation along the z axis. We dilate the coordinates and functions

$$r^2 \to \delta^2\eta,\ z \to z,\ 2ur \to \delta^2 q,\ vr \to \delta A,\ w \to w,\ \rho \to \rho,\ p \to p,\ g \to g.$$

The boundary $r_0(z)$ goes over to $\eta_0(z)$. After substitution, the equations of motion and continuity have the form

$$\begin{gathered} qA_\eta + wA_z = 0, \quad \rho\delta^2(qq_\eta - q^2/(2\eta) + wq_z)/2 - \rho A^2/\eta = -2\eta p_\eta, \\ \rho(qw_\eta + ww_z) = -p_z - \rho g, \quad q_\eta + w_z = 0, \quad q\rho_\eta + w\rho_z = 0 \end{gathered} \tag{1.2}$$

(the subscripts denote differentiation with respect to the appropriate variable). The following boundary conditions are specified at the axis of symmetry and the core boundary

$$q = A = 0 \qquad (\eta = 0); \tag{1.3}$$

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$$p = -1/(2\eta_0) - gz \qquad (\eta = \eta_0); \tag{1.4}$$

$$q = w(\eta_{0z}) \qquad (\eta = \eta_0). \tag{1.5}$$

Condition (1.4) follows from (1.1) and the requirement of continuity of pressure at the core boundary. Equation (1.5) is a kinematic condition.

It is assumed that $\delta << 1$. The terms in (1.2) proportional to δ^2 are neglected. The resultant system is transformed in the same way as in [1]. New independent variables z', ν, $\nu \in [0, 1]$ are introduced in accordance with the relations $z = z'$, $\eta = R(z', \nu)$ where R satisfies

$$wR_{z'} = q \tag{1.6}$$

and the boundary conditions

$$R(z', 0) = 0, \quad R(z', 1) = \eta_0. \tag{1.7}$$

The initial value $R(0, \nu)$ is an arbitrary, single-valued, continuous function satisfying (1.7). Boundary conditions (1.3) (for q) and (1.5) are automatically satisfied for this definition of R. The unknown boundary $\eta_0(z)$ is changed to the known boundary $\nu = 1$. In view of $\delta << 1$ the system (1.2) in the variables z', ν takes on the form [1] (henceforth, the prime on z' is omitted)

$$\begin{gathered} A = A(\nu), \quad \rho = \rho(\nu), \quad (wR_\nu)_z = 0, \\ \rho w w_z = -\Big(\frac{1 - \rho_1 A_1^2}{2R_1^2}\Big) R_{1z} + \Big(\int\limits_\nu^1 \Big[\frac{a(\nu)}{2R}\Big] d\nu\Big)_z + (1-\rho)g, \end{gathered} \tag{1.8}$$

where $a(\nu) = (\rho A^2)_\nu$; R_1, A_1, and ρ_1 are the values of R, A, and ρ with $\nu = 1$ (at the core boundary). System (1.8) is solved with initial data at $z = 0$. It is assumed that $w = w_0(\nu)$ and $R = \nu$ with $z = 0$.

The case $A = 0$, $\rho = \text{const}$ was studied in [1]. In the present paper the results are generalized for $A = A(\nu)$, $\rho = \rho(\nu)$. Natural restrictions imposed on the functions specified by the initial conditions are as follows: $\rho(\nu) \geq \beta > 0$, $w_0(\nu) \geq \gamma > 0$. It is assumed that $\rho(\nu)$, $w_0(\nu)$, $a(\nu)/\nu$ are continuous and bounded in [0, 1]. The boundedness of $a(\nu)/\nu$ as $\nu \to 0$ follows from the requirement of boundedness of the vertical vorticity component at the axis.

Let us denote $x = \rho(w^2 - w_0^2)$, $\xi_0 = \rho^{1/2} w_0$, $y = 2g(1 - \rho)z$. Then after integration with respect to z from 0 to z (1.8) assumes into the form

$$\begin{gathered} y = F(x), \\ F(x) = x - (1 - \rho_1 A_1^2)(R_1^{-1} - 1) - \int\limits_\nu^1 [a(t)R^{-1}]\, dt + \int\limits_\nu^1 [a(t)t^{-1}]\, dt, \\ R(t) = \int\limits_0^t \xi_0(\xi_0^2 + x)^{-1/2}\, du, \quad R_1 = R(1). \end{gathered} \tag{1.9}$$

Thus, the problem is reduced to the solution of nonlinear integral equation (1.9). Below, y is assumed to be the independent variable. We first prove the local solvability of (1.9) in the vicinity $x = 0$, $y = 0$. We then study the behavior of the solution $x(\nu, y)$ as a function of y.

2. Common Properties of Operator F(x). Let C be the space of continuous functions on the segment [0, 1], and D the set of functions x in C such that $x > -\xi_0^2$. It is obvious that D is open in the metric space C.

Proposition 2.1. *F(x) maps D into C. Mapping F(x) is continuously Frèchet differentiable in D, with* $F'(x)h = (I - B_1(x) + B_2(x))h$, *where I is the unit operator;* B_1 *and* B_2 *are bounded linear operators:*

$$B_1(x)h = (1-\rho_1 A_1^2)R_1^{-2}\int_0^1 (\xi_0/2)(\xi_0^2+x)^{-3/2}h\,dt +$$

$$+\int_0^1 a(t)R^{-2}\Big(\int_0^t (\xi_0/2)(\xi_0^2+x)^{-3/2}h\,du\Big)\,dt,$$

$$B_2(x)h = \int_0^\nu a(t)R^{-2}\Big(\int_0^t (\xi_0/2)(\xi_0^2+x)^{-3/2}h\,du\Big)\,dt.$$

Proof. If $x \in D$, then by definition R, R_1, a/R, and a/ν are bounded on the segment $\nu \in [0, 1]$. Then $\forall x \in D$, $y(\nu) = F(x)$ will be a continuous function on [0, 1], hence $F(x) \in C$. We set $F(x + h) - F(x) - F'(x)h = \|h\|\,\omega(\nu, x, h)$; F(x), R and R_1 are expressed in terms of functions having continuous derivatives with respect to x as the argument. For given $x \in D$ the quantities ξ_0, x, h, R, R_1, and a/R will be bounded $\forall\nu \in [0, 1]$. Then by virtue of the continuous dependence of F(x) and F'(x) on x as on their argument and on ν, $\omega(\nu, x, h)$ will be uniformly continuous in its arguments, and $\omega \to 0$ as $\|h\| \to 0$. By virtue of the uniform continuity of ω, ω vanishes uniformly, which proves the existence of F'(x); F'(x) is continuous in D, since it is expressed in terms of functions depending continuously on x as the argument.

We now find the conditions for the linear operator F'(x) to have a bounded inverse. To this end, we first prove

Proposition 2.2. *Operator* $[I + B_2(x)]$ *has a bounded inverse, with*

$$[I + B_2(x)]^{-1}h = \sum_{n=0}^{\infty}(-1)^n B_2^n h, \quad h \in C. \tag{2.1}$$

Proof. For fixed $x \in D$ the following holds: $|\xi_0(\xi_0^2 + x)^{-3/2}/2| \le M_1$, $|taR^{-2}| \le M_2$, M_1, M_2 are constants. Then $|B_2h| \le \|h\|M\nu$, $|B_2^nh| \le \|h\|M^n\nu^n/(n!)^2$, where $M = M_1M_2$. Thus, the series (2.1) is absolutely convergent. The equality $(I + B_2)(I + B_2)^{-1}h = h$ is verified directly from (2.1).

Proposition 2.3. *F'(x) has a limited inverse if*

$$B_1(x)[I + B_2(x)]^{-1}1 \neq 1. \tag{2.2}$$

Proof. From the form of B_1 it follows that $\forall f \in C$ B_1f = const. Then one can verify by direct check that

$$[F'(x)]^{-1}f = ((I+B_2)^{-1}1)(B_1(I+B_2)^{-1}f)(I - B_1(I+B_2)^{-1}1)^{-1} + (I+B_2)^{-1}f, \tag{2.3}$$

whence the proof of the proposition follows.

The following theorem stems from propositions 2.1-2.3

Theorem 2.1. *Equation* (1.9) *is solvable in a certain vicinity of the point* (x_*, y_*), *where* $y_* = F(x_*)$, $x_* \in D$, *if* $B_1(x_*)[I + B_2(x_*)]^{-1}1 \neq 1$. *In this case x is differentiable with respect to y and*

$$x_y'(y_*) = [F'(x_*)]^{-1}. \tag{2.4}$$

Proof. According to propositions 2.1-2.3, the mapping y = F(x) satisfies in the vicinity of $x_* \in D$ the conditions of the implicit function theorem [5]: operator F(x) is continuous in D, $F(x) \in C$, while operator F'(x) exists, is continuous, and has a bounded inverse. Hence follows the proof of the theorem.

3. Structure of the Solutions. Let us emphasize first that it follows from (2.3) and (2.4) that $\forall y$ = const

$$\begin{aligned} &x_y'(\nu, y)y = [F'(x)]^{-1}y = yG(\nu, x)G_1^{-1}(x),\\ &G(\nu, x) = [I + B_2(x)]^{-1}1, \quad G_1(x) = 1 - B_1(x)[I + B_2(x)]^{-1}1. \end{aligned} \tag{3.1}$$

The following proposition establishes qualitatively different behavior of the solutions as functions of the initial data.

Proposition 3.1. *If $G_1(0) > 0$, then for sufficiently small ν $x_y'(\nu, y) > 0$, if $G_1(0) < 0$, then $x_y'(\nu, y) < 0$ for those y for which the inequalities $G_1(x) > 0$ and $G_1(x) < 0$ hold.*

Proof. Since $x = 0 \in D$, $F(0) = 0$, in line with Theorem 2.1, Eq. (1.9) is solvable in the vicinity of the point $x = 0$, $y = 0$, if $G_1(0) \neq 0$. In this case the function $x(\nu, y)$ is differentiable with respect to y, x_y' being expressed by the formula (3.1). Obviously, (1.9) is solvable as long as $G_1(0) \neq 0$ or until x reaches the boundary of the domain D. From the proof of proposition 2.2 it follows that $G(\nu, x) > 0$ for sufficiently small ν. Accordingly, if $G_1(x) > 0$, then $x_y' > 0$; if $G_1(x) < 0$, then $x_y' < 0$ for sufficiently small ν.

Thus, as y increases, x either increases or decreases in the vicinity of the vortex axis as a function of the initial data.

Further propositions are proved under additional restrictions $a(\nu) > 0$, $\rho_1 A_1^2 < 1$. The restrictions are natural, since the flows in which they are violated possess centrifugal Rayleigh instability.

In this case in order for the inequality $G_1(0) > 0$ to be valid it is sufficient for a more evident relationship to hold.

Proposition 3.1a. *If $a(\nu) > 0$, $\rho_1 A_1^2 < 1$, $B_1(0)1 < 1$, then $G_1(0) > 0$.*

Proof. The inequalities $0 < B_2(0)1 < B_1(0)1 < 1$ follow from the above assumptions and the form of B_1 and B_2. Applying the operator $B_1 B_2^n$ to the inequalities $0 < B_2(0)1 < 1$, we obtain $0 < B_1 B_2^{n+1} 1 < B_1 B_2^n 1$. Hence, from (2.1) and the definition of $G_1(x)$ it follows that $G_1(0)$ is represented by an alternating series with monotonically vanishing terms. The sum of the series is not smaller than the difference of the first two terms. Then $G_1(0) > 1 - B_1(0)1 > 0$.

It has been established in Proposition 3.1 that as y increases, x increases or decreases in the vicinity of the axis as a function of the initial data with $y = 0$. Below is established how the dependence of x on ν changes qualitatively in the vicinity of the axis as y increases.

Corollary 3.1. *Let $a(\nu) > 0$, $\rho_1 A_1^2 < 1$. If $G_1(x) > 0$, then $x_{\nu y}'' < 0$; if $G_1(x) < 0$, then $x_{\nu y}'' > 0$ for sufficiently small ν.*

Proof. Following (3.1), $x_{\nu y}''(\nu, y) = G_\nu'(\nu, x) G_1^{-1}(x)$. It follows from the expression for $G(\nu, x)$ that

$$G_\nu'(\nu, x) = -a(\nu) R^{-2} \int_0^\nu (\xi_0/2)(\xi_0^2 + x)^{-3/2} G(t, x)\, dt. \tag{3.2}$$

It was noted in Proposition 3.1 that $G(\nu, x) > 0$ for sufficiently small ν. Then in view of the positiveness of a and ξ_0 it follows from (3.2) that $G_\nu' < 0$ for sufficiently small ν. In view of the expression for $x_{\nu y}''$, hence follows the proof of the corollary.

Thus, if $G_1(x) > 0$, then x increases with increasing y in the vicinity of the axis, the greatest increase of x occurring at the axis. If $G_1(x) < 0$, x vanishes in the vicinity of the axis, with the maximum decrease at the axis.

The following proposition establishes a condition whose fulfillment leads to the nonexistence of a solution of Eq. (1.9) with finite $y > 0$.

Proposition 3.2. *If $a(\nu) > 0$, $\rho_1 A_1^2 < 1$, $G_1(0) < 0$, then Eq. (1.9) has no solution for finite $y > 0$.*

Proof. If $G_1(0) < 0$, then, in line with Theorem 2.1, Eq. (1.9) is solvable in the vicinity of the point $x = 0$, $y = 0$ and has the solution $x(\nu, y)$. Obviously, the solution can be continued as long as $G_1(x) \neq 0$ or until x attains the boundary of the domain D. If the expression $G_1(x)$ vanishes for a certain finite y, the proposition is proved, since, in line with Theorem 2.1, Eq. (1.9) has no solution in this case. Let us assume that $G_1(x)$ does not vanish. Then, in view of the condition $G_1(0) < 0$ we have $G_1(x) < 0$. It follows from the definitions of $G(\nu, x)$ and $B_2(x)$ and from (2.1) that $G(0, x) = 1$. We then obtain $x_y'(0, y) = G_1^{-1}(x) < 0$ from (3.1). Hence, with allowance for the equality $x = 0$ at $y = 0$ it follows that $x(0, y) < 0$ for $y > 0$. In view of the positiveness of R, a, and $1 - \rho_1 A_1^2$ we obtain from (1.9) with $\nu = 0$ the inequality

$$y \le (1 - \rho_1 A_1^2) + \int_0^1 a(t) t^{-1}\, dt. \tag{3.3}$$

Thus, y is bounded.

Let us consider the special case $\rho_1 A_1^2 = 1$. This equation holds, for example, when there is no jump of density and rotational velocity component at the core boundary ($\nu = 1$). In this case the results of propositions 3.1 and 3.2 can be extended.

Corollary 3.2. *Let $\rho_1 A_1^2 = 1$, $a(\nu) > 0$. If $G_1(0) < 0$, then for those x for which the inequality $G_1(x) < 0$ is valid $x_y'(\nu, y) < 0$ for sufficiently small ν and $x_y'(\nu, y) > 0$ for ν sufficiently close to unity.*

Proof. Inequality $x_y' < 0$ is proved for sufficiently small ν in Proposition 3.1. Let us prove the second part. If $\rho_1 A_1^2 = 1$, it is obvious that $B_1(x)$ equals the value of $B_2(x)$ with $\nu = 1$. It then follows from (2.1) and (3.1) that $G_1(x) = G(1, x)$ in this case. Hence, taking account of (3.1) we obtain

$$x_y'(\nu, y) = G(\nu, x)G^{-1}(1, x), \tag{3.4}$$

where x_y' is continuous in ν, since $G(\nu, x)$ is represented by an absolutely convergent series whose terms are continuous in ν. It follows from (3.4) that $x_y'(1, y) = 1$. In view of the continuity of x_y' in ν the inequality $x_y'(\nu, y) > 0$ is also held for those ν which are sufficiently close to one.

Thus, in this case x decreases near the axis and increases near the core boundary.

Proposition 3.3. *If $\rho_1 A_1^2 = 1$, Eq. (1.9) has no solution with finite $y < 0$.*

Proof. It follows from (1.9) that $y = x$ with $\nu = 1$. Since $x > -\xi_0^2$, we have $y > -\xi_0^2(1)$.

4. Discussion of Results. The motion of a fluid in the core of a vertical tornado-like vortex core has been studied previously [1] without regard for its rotation about the axis. It has been shown that in this case the vertical velocity component in the core varies as a function of z for the same value with any ν, i.e., the profile of the vertical velocity component as a function of ν for fixed z does not deform during motion. However, to construct a model of the decay of a vortex [1], it was most important that the vertical velocity component have a minimum at the axis, so that the stagnation point in the process of motion would appear at the vortex axis. Therefore, it was assumed that such a minimum occurs on the initial profile $w_0(\nu)$. In this connection the possibility of deformation of the profile during motion and the type of the deformation are important issues for understanding the mechanisms of the inception of such phenomena as the vortex decay or jump.

On the basis of the results we analyze what changes take place when fluid rotation in the core is taken into account. For a more obvious presentation we let $\rho = \rho_1 = \text{const}$. It should be noted that this assumption does not change the qualitative conclusions obtained below. We consider the case when $\rho_1 < 1$, i.e., when the fluid in the core is lighter than the surrounding fluid. Such flows are realized in natural vortices [4], as well as in laboratory vortices [3, 6, 7].

Since $\rho = \text{const}$, it follows that the condition $z = \text{const}$ corresponds to $y = \text{const}$. Since $\rho_1 < 1$, the increase of z corresponds to the increase of y. Therefore, the above results can be used directly. For example, if $G_1(0) > 0$, the vertical velocity component in the vicinity of the vortex axis grows with the increase in height; if $G_1(0) < 0$, it decreases (Proposition 3.1), with the greatest rate of decrease at the axis. Thus, the profile of the vertical velocity component is deformed as the height increases, and the stagnation point can occur at the axis only if initial data with $z = 0$ are such that $G_1(0) < 0$. Moreover, in this case, according to proposition 3.2, there exists no solution with a certain finite $z > 0$. One can assume as in [1] that the nonexistence of solution suggests the decay of the vortex. Then $G_1(0) < 0$ is the condition of decay, and the restrictions on the vortex height can be obtained using (3.3).

Let us carry out qualitative estimates in dimensional form. Assuming that the fluid in the vortex core rotates according to a rigid rotation law (which is consistent with observations [2, 4]), we obtain $A = (v_1/v_0)\nu$. Here v_1 and v_0 are the rotational velocity components at $z = 0$ and $r = r_0$ in the core and outer flow, respectively (in general, $v_1 \neq v_0$ if we introduce a discontinuity of the rotational velocity component at the core boundary). By the definition of y, inequality (3.3) has the form

$$z \leq \frac{\rho_0 v_0^2 + \rho_1 v_1^2}{2g(\rho_0 - \rho_1)} \tag{4.1}$$

(ρ_1 and ρ_0 are the densities in the vortex core and in the outer flow, $\rho_1 < \rho_0$).

We find an expression for $G_1(0)$, assuming that $w_0 = \text{const}$. This assumption well approximates the experimental data of [2]. Then

$$G_1(0) = \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n!)^2} - \frac{\rho_0 v_0^2 - \rho_1 v_1^2}{2\rho_1 w_0^2} \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n!)^2 (n+1)},$$

where $M = (v_1/w_0)^2$. Normally at the interface of the core and the outer flow $\rho_0 v_0^2 \cong \rho_1 v_1^2$. Then

$$G_1(0) \cong \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n!)^2}.$$

Hence $G_1(0) > 0$ if $M < 1.4$. Thus, the vertical velocity component in the vicinity of the vortex axis will increase if the inequality $(v_1/w_0)^2 < 1.4$ is valid.

By hypothesis, the nonexistence of a solution for finite z is the condition for inception of vortex decay. The condition is satisfied if $G_1(0) < 0$. The vortex core observed in the experiments [3, 7] and in nature [8] is of finite height. Therefore, if the assumption is true, theoretical estimates of the vortex height and $G_1(0)$ should agree with the measurement results [3, 7, 8].

To within the coefficient 1/2, Eq. (4.1) is analogous to the expression obtained in [1] for the vortex height without regard for fluid rotation in the core. Therefore, the height calculated from (4.1) agrees in order of magnitude with the observation results in [3, 7, 8].

The quantity $G_1(0)$ can be calculated from [7]. We assume that $v_1 \cong v_0 = 100$ cm/sec, $w_0 = 40$-50 cm/sec. For $M = 4; 6; 7$ we have $G_1(0) \cong -0.35; -0.2; -0.05 < 0$.

The features of the motion of a swirled fluid in the core of a vertical, tornado-like vortex has been analyzed. A rigorous criterion for identifying qualitatively different behavior of the solutions as a function of the initial data has been revealed. Analytical estimates agree in the order of magnitude with the observation results for laboratory and natural vortices. A theoretical basis has been proposed for numerical calculations of the fluid flow in the cores of vertical tornado-like vortices.

REFERENCES

1. V. V. Nikulin, "Decay of a vertical, tornado-like vortex," Prikl. Mekh. Tekh. Fiz., No. 4, 42-47 (1992).
2. V. V. Nikulin, "Interaction of a tornado-like vortex with rigid boundaries," Prikl. Mekh. Tekh. Fiz., No. 1, 68-75 (1980).
3. J. B. Mullen and T. Maxworthy, "A laboratory model of dust devil vortices," Dynamics Atmos. and Oceans, No. 1, 181-214, (1977).
4. P. C. Sinclair, "The lower structure of dust devils," J. Atmospheric Sci., **30**, No. 8, 1599-1619 (1973).
5. A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis [in Russian], Nauka, Moscow (1976).
6. V. V. Nikulin, "Experimental measurements of temperature in a tornado-like vortex," Continuum Dynamics [in Russian], Novosibirsk, Vol. 6 (1983).
7. D. E. Fitzjarrald, "A laboratory simulation of convective vortices," J. Atmospheric Sci., **30**, No. 7, 894-902 (1973).
8. N. R. Williams, "Development of dust whirls," Bull. Am. Meteorol. Soc., **29**, No. 3, 106-117 (1948).